INTRODUCTION TO ARBITRAGE PRICING OF FINANCIAL DERIVATIVES*

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1 Call and Put Spot Options

Let us first describe briefly the set of general assumptions imposed on our models of financial markets. We consider throughout, unless explicitly stated otherwise, the case of a so-called *frictionless market,* meaning that: all investors are price-takers, all parties have the same access to the relevant information, there are no transaction costs or commissions, and all assets are assumed to be perfectly divisible and liquid. There is no restriction whatsoever on the size of a bank credit, and the lending and borrowing rates are equal. Finally, individuals are allowed to sell short any security and receive full use of the proceeds (of course, restitution is required for payoffs made to securities held short). Unless otherwise specified, by an *option* we shall mean throughout a European option, giving the right to exercise the option only at the expiry date. In mathematical terms, the problem of pricing of American options is closely related to *optimal stopping* problems. Unfortunately, closed-form expressions for the prices of American options are rarely available; for instance, no closed-form solution is available for the price of an American put option in the now classic framework of the Black-Scholes option pricing model.

A *European call option* written on a common stockis a financial security that gives its holder the right (but not the obligation) to buy the underlying stock on a prespecified date and for a prespecified price. The act of making this transaction is referred to as *exercising* the option. If an option is not exercised, we say it is *abandoned.* Another class of options comprises so-called *American options.* These may be exercised at any time on or before the prespecified date. The prespecified fixed price, say K, is termed the *strike* or *exercise* price; the terminal date, denoted by T in what follows, is called the *expiry date* or *maturity.* It should be emphasized that an option gives the holder the right to do something; however, the holder is not obliged to exercise this right. In order to purchase an option contract, an investor needs to pay an option's price (or *premium*) to a second party at the initial date when the contract is entered into.

Let us denote by S_T the stock price at the terminal date T. It is natural to assume that S_T is not known at time 0, hence S_T gives rise to uncertainty in our model. We argue that from the perspective of the option holder, the payoff g at expiry date T from a European call option is given by the formula

$$g(S_T) = (S_T - K)^+ \stackrel{\text{def}}{=} \max\{S_T - K, 0\}, \tag{1}$$

*The present text is based on Chapter I of the monograph: M. Musiela and M. Rutkowski: *Martingale Methods in Financial Modelling.* Springer-Verlag, Heidelberg Berlin New York, 1997 that is

$$g(S_T) = \begin{cases} S_T - K & \text{if } S_T > K \text{ (option is exercised)}, \\ 0 & \text{if } S_T \le K \text{ (option is abandoned)}. \end{cases}$$

In fact, if at the expiry date T the stock price is lower than the strike price, the holder of the call option can purchase an underlying stock directly on a spot (i.e., cash) market, paying less than K. In other words, it would be irrational to exercise the option, at least for an investor who prefers more wealth to less. On the other hand, if at the expiry date the stock price is greater than K, an investor should exercise his right to buy the underlying stock at the strike price K. Indeed, by selling the stock immediately at the spot market, the holder of the call option is able to realize an instantaneous net profit $S_T - K$ (note that transaction costs and/or commissions are ignored here). In contrast to a call option, a *put option* gives its holder the right to sell the underlying asset by a certain date for a prespecified price. Using the same notation as above, we arrive at the following expression for the payoff h at maturity T from a European put option

$$h(S_T) = (K - S_T)^+ \stackrel{\text{def}}{=} \max\{K - S_T, 0\}, \tag{2}$$

or, more explicitly,

$$h(S_T) = \begin{cases} 0 & \text{if } S_T \ge K \text{ (option is abandoned)}, \\ K - S_T & \text{if } S_T < K \text{ (option is exercised)}. \end{cases}$$

It follows immediately that the payoffs of call and put options satisfy the following simple but useful equality

$$g(S_T) - h(S_T) = (S_T - K)^+ - (K - S_T)^+ = S_T - K. \tag{3}$$

The last equality can be used, in particular, to derive the so-called *put-call parity* relationship for option prices. Basically, put-call parity means that the price of a European put option is determined by the price of a European call option with the same strike and expiry date, the current price of the underlying asset, and the properly discounted value of the strike price.

1.1 One-period Spot Market

Let us start by considering an elementary example of an option contract.

Example 1.1 Assume that the current stock price is \$280, and after three months the stock price may either rise to \$320 or decline to \$260. We shall find the rational price of a 3-month European call option with strike price $K = \$280$, provided that the simple risk-free interest rate r for 3-month deposits and loans is $r = 5\%$.

Suppose that the subjective probability of the price rise is 0.2, and that of the fall is 0.8; these assumptions correspond, loosely, to a so-called *bear market*. Note that the word *subjective* means that we take the point of view of a particular individual. Generally speaking, the two parties involved in an option contract may have (and usually do have) differing assessments of these probabilities. To model a *bull market* one may assume, for example, that the first probability is 0.8, so that the second is 0.2.

Let us focus first on the bear market case. The terminal stock price S_T may be seen as a random variable on a probability space $\Omega = \{\omega_1, \omega_2\}$ with a probability measure $\mathbf{P}$ given by

$$\mathbf{P}\{\omega_1\} = 0.2 = 1 - \mathbf{P}\{\omega_2\}.$$

Formally, S_T is a function $S_T : \Omega \to R_+$ given by the following formula

$$S_T(\omega) = \begin{cases} S^u = 320, & \text{if } \omega = \omega_1, \\ S^d = 260, & \text{if } \omega = \omega_2. \end{cases}$$

Consequently, the terminal option's payoff $X = C_T = (S_T - K)^+$ satisfies

$$C_T(\omega) = \begin{cases} C^u = 40, & \text{if } \omega = \omega_1, \\ C^d = 0, & \text{if } \omega = \omega_2. \end{cases}$$

Note that the expected value under $\mathbf{P}$ of the discounted option's payoff equals

$$\mathbf{E}_{\mathbf{P}}\left((1+r)^{-1}C_T\right) = 0.2 \times 40 \times (1.05)^{-1} = 7.62.$$

It is clear that the above expectation depends on the choice of the probability measure $\mathbf{P}$; that is, it depends on the investor's assessment of the market. For a call option, the expectation corresponding to the case of a bull market would be greater than that which assumes a bear market. In our example, the expected value of the discounted payoff from the option under the bull market hypothesis equals 30.48. Still, to construct a reliable model of a financial market, one has to guarantee the uniqueness of the price of any derivative security. This can be done by applying the concept of the so-called replicating portfolio, which we will now introduce.

1.2 Replicating Portfolios

The two-state option pricing model presented below was developed independently by Sharpe (1978) and Rendleman and Bartter (1979) (a point worth mentioning is that the ground-breaking papers of Black and Scholes (1973) and Merton (1973), who examined the arbitrage pricing of options in a continuous-time framework, were published five years earlier). The idea is to construct a portfolio at time 0 which replicates exactly the option's terminal payoff at time T. Let $\phi = \phi_0 = (\alpha_0, \beta_0) \in \mathbf{R}^2$ denote a portfolio of an investor with a short position in one call option. More precisely, let α_0 stand for the number of shares of stock held at time 0, and β_0 be the amount of money deposited on a bank account or borrowed from a bank. By $V_t(\phi)$ we denote the wealth of this portfolio at dates $t = 0$ and $t = T$; that is, the payoff from the portfolio ϕ at given dates. It should be emphasized that once the portfolio is set up at time 0, it remains fixed until the terminal date T. Therefore, for its wealth process $V(\phi)$ we have

$$V_0(\phi) = \alpha_0 S_0 + \beta_0 \quad \text{and} \quad V_T(\phi) = \alpha_0 S_T + \beta_0(1+r). \tag{4}$$

We say that a portfolio ϕ *replicates* the option's terminal payoff whenever $V_T(\phi) = C_T$, that is, if

$$V_T(\phi)(\omega) = \begin{cases} V^u(\phi) = \alpha_0 S^u + (1+r)\beta_0 = C^u, & \text{if } \omega = \omega_1, \\ V^d(\phi) = \alpha_0 S^d + (1+r)\beta_0 = C^d, & \text{if } \omega = \omega_2. \end{cases}$$

For the data of Example 1.1, the portfolio ϕ is determined by the following system of linear equations

$$\begin{cases} 320\,\alpha_0 + 1.05\,\beta_0 = 40, \\ 260\,\alpha_0 + 1.05\,\beta_0 = 0, \end{cases}$$

with unique solution $\alpha_0 = 2/3$ and $\beta_0 = -165.08$. Observe that for every call we are short, we hold α_0 of stock[1] and the dollar amount β_0 in riskless bonds in the hedging portfolio. Put another way, by purchasing shares and borrowing against them in the right proportion, we are able to replicate an option position. (Actually, one can easily check that this property holds for any *contingent claim* X which settles at time T.) It is natural to define the *manufacturing cost* C_0 of a call option as the initial investment needed to construct a replicating portfolio, i.e.,

$$C_0 = V_0(\phi) = \alpha_0 S_0 + \beta_0 = (2/3) \times 280 - 165.08 = 21.59.$$

It should be stressed that in order to determine the manufacturing cost of a call we did not need to know the probability of the rise or fall of the stock price. In other words, it appears that

[1] We shall refer to the number of shares held for each call sold as the *hedge ratio*. Basically, to *hedge* means to reduce risk by making transactions that reduce exposure to market fluctuations.

the manufacturing cost is invariant with respect to individual assessments of market behavior. In particular, it is identical under the bull and bear market hypotheses. To determine the *rational* price of a call we have used the option's strike price, the current value of the stock price, the range of fluctuations in the stock price (that is, the future levels of the stock price), and the risk-free rate of interest. The investor's transactions and the corresponding cash flows may be summarized by the following two exhibits

$$\text{at time } t=0 \quad \begin{cases} \text{one written call option} & C_0, \\ \alpha_0 \text{ shares purchased} & -\alpha_0 S_0, \\ \text{amount of cash borrowed} & \beta_0, \end{cases}$$

and

$$\text{at time } t=T \quad \begin{cases} \text{payoff from the call option} & -C_T, \\ \alpha_0 \text{ shares sold} & \alpha_0 S_T, \\ \text{loan paid back} & -\hat{r}\beta_0, \end{cases}$$

where $\hat{r} = 1+r$. It should be observed that no net initial investment is needed to establish the above portfolio; that is, the portfolio is costless. On the other hand, for each possible level of stock price at time T, the hedge exactly breaks even on the option's expiry date. Also, it is easy to verify that if the call were not priced at \$21.59, it would be possible for a sure profit to be gained, either by the option's writer (if the option's price were greater than its manufacturing cost) or by its buyer (in the opposite case). Still, the manufacturing cost cannot be seen as a fair price of a claim X, unless the market model is *arbitrage-free,* in a sense examined below. Indeed, it may happen that the manufacturing cost of a non-negative claim is a strictly negative number. Such a phenomenon contradicts the usual assumption that it is not possible to make riskless profits.

1.3 Martingale Measure for a Spot Market

Although, as shown above, subjective probabilities are useless when pricing an option, probabilistic methods play an important role in contingent claims valuation. They rely on the notion of a *martingale,* which is, intuitively, a probabilistic model of a fair game. In order to apply the so-called *martingale method* of derivative pricing, one has to find first a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$, and such that the *discounted* (or *relative*) stock price process S^*, which is defined by the formula

$$S_0^* = S_0, \quad S_T^* = (1+r)^{-1} S_T,$$

follows a $\mathbf{P}^*$-martingale; that is, the equality $S_0^* = \mathbf{E}_{\mathbf{P}^*}(S_T^*)$ holds. Such a probability measure $\mathbf{P}^*$ is called a *martingale measure* for the discounted stock price process S^*. In the case of a two-state model, the probability measure $\mathbf{P}^*$ is easily seen to be uniquely determined (provided it exists) by the following linear equation

$$S_0 = (1+r)^{-1}(p_* S^u + (1-p_*)S^d), \tag{5}$$

where $p_* = \mathbf{P}^*\{\omega_1\}$ and $1 - p_* = \mathbf{P}^*\{\omega_2\}$. Solving this equation for p_* yields

$$\mathbf{P}^*\{\omega_1\} = \frac{(1+r)S_0 - S^d}{S^u - S^d}, \quad \mathbf{P}^*\{\omega_2\} = \frac{S^u - (1+r)S_0}{S^u - S^d}. \tag{6}$$

Let us now check that the price C_0 coincides with C_0^*, where we write C_0^* to denote the expected value under $\mathbf{P}^*$ of an option's discounted terminal payoff – that is

$$C_0^* \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}C_T\big) = \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}(S_T - K)^+\big).$$

Indeed, using the data of Example 1.1 we find $p_* = 17/30$, so that

$$C_0^* = (1+r)^{-1}\big(p_* C^u + (1-p_*)C^d\big) = 21.59 = C_0.$$

Remarks. Observe that since the process S^* follows a $\mathbf{P}^*$-martingale, we may say that the discounted stock price process may be seen as a fair game model in a *risk-neutral economy* – that is, in the stochastic economy in which the probabilities of future stock price fluctuations are determined by the martingale measure $\mathbf{P}^*$. It should be stressed, however, that the fundamental idea of arbitrage pricing is based solely on the existence of a portfolio that hedges perfectly the risk exposure related to uncertain future prices of risky securities. Therefore, the probabilistic properties of the model are not essential. In particular, we do not assume that the real-world economy is actually risk-neutral. On the contrary, the notion of a risk-neutral economy should be seen rather as a technical tool. The aim of introducing the martingale measure is twofold: first, it simplifies the explicit evaluation of arbitrage prices of derivative securities. Second, it describes the arbitrage-free property of a given pricing model for primary securities in terms of the behavior of relative prices. This approach is frequently referred to as the *partial equilibrium approach,* as opposed to the *general equilibrium approach.* Let us stress that in the latter theory the investors' preferences, usually described in stochastic models by means of their (expected) utility functions, play an important role.

To summarize, the notion of an arbitrage price for a derivative security does not depend on the choice of a probability measure in a particular pricing model for primary securities. More precisely, using standard probabilistic terminology, this means that the arbitrage price depends on the support of a subjective probability measure $\mathbf{P}$, but is invariant with respect to the choice of a particular probability measure from the class of mutually equivalent probability measures. In financial terminology, this can be restated as follows: all investors agree on the range of future price fluctuations of primary securities; they may have different assessments of the corresponding subjective probabilities, however.

1.4 Absence of Arbitrage

Let us consider a simple two-state, one-period, two-security market model defined on a probability space $\Omega = \{\omega_1, \omega_2\}$ equipped with the σ-fields $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = 2^\Omega$ (i.e., $\mathcal{F}_T$ contains all subsets of Ω), and a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbf{P}\{\omega_1\}$ and $\mathbf{P}\{\omega_2\}$ are strictly positive numbers. The first security is a stock whose price process is modelled as a strictly positive discrete-time process $S = (S_t)_{t\in\{0,T\}}$. We assume that the process S is $(\mathcal{F}_t)$-adapted, i.e., that the random variables S_t are $\mathcal{F}_t$-measurable for $t \in \{0, T\}$. This means that S_0 is a real number, and

$$S_T(\omega) = \begin{cases} S^u & \text{if } \omega = \omega_1, \\ S^d & \text{if } \omega = \omega_2, \end{cases}$$

where, without loss of generality, $S^u > S^d$. The second security is a riskless bond whose price process is $B_0 = 1$, $B_T = 1 + r$ for some real $r \geq 0$. Let Φ stand for the linear space of all stock-bond portfolios $\phi = \phi_0 = (\alpha_0, \beta_0)$, where α_0 and β_0 are real numbers (clearly, the class Φ may be thus identified with $\mathbf{R}^2$). We shall consider the pricing of contingent claims in a security market model $\mathcal{M} = (S, B, \Phi)$. We shall now check that an arbitrary *contingent claim* X which settles at time T (i.e., any $\mathcal{F}_T$-measurable real-valued random variable) admits a unique replicating portfolio in our market model. In other words, an arbitrary contingent claim X is *attainable* in the market model $\mathcal{M}$. Indeed, if

$$X(\omega) = \begin{cases} X^u & \text{if } \omega = \omega_1, \\ X^d & \text{if } \omega = \omega_2, \end{cases}$$

then the replicating portfolio ϕ is determined by the following system of linear equations

$$\begin{cases} \alpha_0 S^u + (1+r)\beta_0 = X^u, \\ \alpha_0 S^d + (1+r)\beta_0 = X^d, \end{cases} \tag{7}$$

which admits a unique solution

$$\alpha_0 = \frac{X^u - X^d}{S^u - S^d}, \quad \beta_0 = \frac{X^d S^u - X^u S^d}{(1+r)(S^u - S^d)}, \tag{8}$$

for arbitrary values of X^u and X^d. Consequently, an arbitrary contingent claim X admits a unique *manufacturing cost* $\pi_0(X)$ in $\mathcal{M}$ which is given by the formula

$$\pi_0(X) \stackrel{\text{def}}{=} V_0(\phi) = \alpha_0 S_0 + \beta_0 = \frac{X^u - X^d}{S^u - S^d} S_0 + \frac{X^d S^u - X^u S^d}{(1+r)(S^u - S^d)}. \tag{9}$$

As already mentioned, the manufacturing cost of a strictly positive contingent claim may appear to be a negative number, in general. If this were the case, there would be a profitable riskless trading strategy (so-called *arbitrage opportunity*) involving only the stock and riskless borrowing and lending. To exclude such situations, which are clearly inconsistent with any broad notion of a rational market equilibrium (as it is common to assume that investors are *non-satiated,* meaning that they prefer more wealth to less), we have to impose further essential restrictions on our market model.

Definition 1.1 We say that a security pricing model $\mathcal{M}$ is *arbitrage-free* if there is no portfolio $\phi \in \Phi$ for which

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0 \quad \text{and} \quad \mathbf{P}\{V_T(\phi) > 0\} > 0. \tag{10}$$

A portfolio ϕ for which the set (10) of conditions is satisfied is called an *arbitrage opportunity.* A *strong arbitrage opportunity* is a portfolio ϕ for which

$$V_0(\phi) < 0 \quad \text{and} \quad V_T(\phi) \geq 0. \tag{11}$$

It is customary to take either (10) or (11) as the definition of an arbitrage opportunity. Note, however, that both notions are not necessarily equivalent. We are in a position to introduce the notion of an arbitrage price; that is, the price derived using the no-arbitrage arguments.

Definition 1.2 Suppose that the security market $\mathcal{M}$ is arbitrage-free. Then the manufacturing cost $\pi_0(X)$ is called the *arbitrage price* of X at time 0 in security market $\mathcal{M}$.

As the next result shows, under the absence of arbitrage in a market model, the manufacturing cost may be seen as the unique rational price of a given contingent claim – that is, the unique price compatible with any rational market equilibrium.

Proposition 1.1 *Suppose that the spot market $\mathcal{M} = (S, B, \Phi)$ is arbitrage-free. Let H stand for the rational price process of some attainable contingent claim X; more explicitly, $H_0 \in \mathbf{R}$ and $H_T = X$. Let Φ_H denote the class of all portfolios in stock, bond and derivative security H. Then the spot market (S, B, H, Φ_H) is arbitrage-free if and only if $H_0 = \pi_0(X)$.*

Proof. Since the proof is straightforward, it is left to the reader. □

1.5 Optimality of Replication

Let us show that replication is, in a sense, an optimal way of hedging. Firstly, we say that a portfolio ϕ *perfectly hedges* against X if $V_T(\phi) \geq X$, that is, whenever

$$\begin{cases} \alpha_0 S^u + (1+r)\beta_0 \geq X^u, \\ \alpha_0 S^d + (1+r)\beta_0 \geq X^d. \end{cases} \tag{12}$$

The minimal initial cost of a perfect hedging portfolio against X is called the *seller's price* of X, and it is denoted by $\pi_0^s(X)$. Let us check that $\pi_0^s(X) = \pi_0(X)$. By denoting $c = V_0(\phi)$, we may rewrite (12) as follows

$$\begin{cases} \alpha_0(S^u - S_0(1+r)) + c(1+r) \geq X^u, \\ \alpha_0(S^d - S_0(1+r)) + c(1+r) \geq X^d. \end{cases} \tag{13}$$

It is trivial to check that the minimal $c \in \mathbf{R}$ for which (13) holds is actually that value of c for which inequalities in (13) become equalities. This means that the replication appears to be the least expensive way of perfect hedging for the seller of X. Let us now consider the other party of the contract, i.e., the buyer of X. Since the buyer of X can be seen as the seller of $-X$, the associated problem is to minimize $c \in \mathbf{R}$, subject to the following constraints

$$\begin{cases} \alpha_0(S^u - S_0(1+r)) + c(1+r) \geq -X^u, \\ \alpha_0(S^d - S_0(1+r)) + c(1+r) \geq -X^d. \end{cases}$$

It is clear that the solution to this problem is $\pi^s(-X) = -\pi(X) = \pi(-X)$, so that replication appears to be optimal for the buyer also. We conclude that the least price the seller is ready to accept for X equals the maximal amount the buyer is ready to pay for it. If we define the *buyer's price* of X, denoted by $\pi_0^b(X)$, by setting $\pi_0^b(X) = -\pi_0^s(-X)$, then

$$\pi_0^s(X) = \pi_0^b(X) = \pi_0(X);$$

that is, all prices coincide. This shows that in a two-state, arbitrage-free model, the arbitrage price of any contingent claim can be defined using the optimality criterion. It appears that such an approach to arbitrage pricing can be extended to other models; we prefer, however, to define the arbitrage price as that value of the price which excludes arbitrage opportunities. Indeed, the fact that observed market prices are close to arbitrage prices predicted by a suitable stochastic model should be explained by the presence of the traders known as *arbitrageurs*[2] on financial markets, rather than by the clever investment decisions of most market participants.

The next proposition explains the role of the so-called *risk-neutral* economy in arbitrage pricing of derivative securities. Observe that the important role of risk preferences in classic equilibrium asset pricing theory is left aside in the present context. Notice, however, that the use of a martingale measure $\mathbf{P}^*$ in arbitrage pricing corresponds to the assumption that all investors are risk-neutral, meaning that they do not differentiate between all riskless and risky investments with the same expected rate of return. The arbitrage valuation of derivative securities is thus done as if an economy actually were *risk-neutral.* Formula (14) shows that the arbitrage price of a contingent claim X can be found by first modifying the model so that the stock earns at the riskless rate, and then computing the expected value of the discounted claim (to the best of our knowledge, this method of computing the price was discovered in Cox and Ross (1976)).

Proposition 1.2 *The spot market $\mathcal{M} = (S, B, \Phi)$ is arbitrage-free if and only if the discounted stock price process S^* admits a martingale measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$. In this case, the arbitrage price at time 0 of any contingent claim X which settles at time T is given by the risk-neutral valuation formula*

$$\pi_0(X) = \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}X\big), \tag{14}$$

or explicitly

$$\pi_0(X) = \frac{S_0(1+r) - S^d}{S^u - S^d}\,\frac{X^u}{1+r} + \frac{S^u - S_0(1+r)}{S^u - S^d}\,\frac{X^d}{1+r}. \tag{15}$$

Proof. We know already that the martingale measure for S^* equivalent to $\mathbf{P}$ exists if and only if the unique solution p_* of equation (5) satisfies $0 < p_* < 1$. Suppose there is no equivalent martingale measure for S^*; for instance, assume that $p_* \geq 1$. Our aim is to construct explicitly an arbitrage opportunity in the market model (S, B, Φ). To this end, observe that the inequality $p_* \geq 1$ is equivalent to $(1+r)S_0 \geq S^u$ (recall that S^u is always greater than S^d). The portfolio $\phi = (-1, S_0)$ satisfies $V_0(\phi) = 0$ and

$$V_T(\phi) = \begin{cases} -S^u + (1+r)S_0 \geq 0 & \text{if } \omega = \omega_1, \\ -S^d + (1+r)S_0 > 0 & \text{if } \omega = \omega_2, \end{cases}$$

so that ϕ is indeed an arbitrage opportunity. On the other hand, if $p_* \leq 0$, then the inequality $S^d \geq (1+r)S_0$ holds, and it is easily seen that in this case the portfolio $\psi = (1, -S_0) = -\phi$ is an

[2]An *arbitrageur* is that market participant who consistently uses the price discrepancies to make (almost) risk-free profits. Arbitrageurs are relatively few, but they are far more active than most long-term investors.

arbitrage opportunity. Finally, if $0 < p_* < 1$ for any portfolio ϕ satisfying $V_0(\phi) = 0,$ then by virtue of (9) and (6) we get

$$p_* V^u(\phi) + (1 - p_*) V^d(\phi) = 0$$

so that $V^d(\phi) < 0$ when $V^u(\phi) > 0$ and $V^d(\phi) > 0$ if $V^u(\phi) < 0$. This shows that there are no arbitrage opportunities in $\mathcal{M}$ when $0 < p_* < 1$. To prove formula (14) it is enough to compare it with (9). Alternatively, we may observe that for the unique portfolio $\phi = (\alpha_0, \beta_0)$ which replicates the claim $X,$ we have

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}X\big) &= \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}V_T(\phi)\big) = \mathbf{E}_{\mathbf{P}^*}(\alpha_0 S_T^* + \beta_0) \\ &= \alpha_0 S_0^* + \beta_0 = V_0(\phi) = \pi_0(X), \end{aligned}$$

so that we are done. □

Remarks. The choice of the bond price process as a discount factor is not essential. Suppose, on the contrary, that we have chosen the stock price S as a *numeraire.* In other words, we now consider the bond price B discounted by the stock price S

$$B_t^* = B_t / S_t$$

for $t \in \{0, T\}$. The martingale measure $\bar{\mathbf{P}}$ for the process B^* is determined by the equality $B_0^* = \mathbf{E}_{\bar{\mathbf{P}}}(B_T^*),$ or explicitly

$$\bar{p}\,\frac{1+r}{S^u} + \bar{q}\,\frac{1+r}{S^d} = \frac{1}{S_0}\,, \tag{16}$$

where $\bar{q} = 1 - \bar{p}$. One finds that

$$\bar{\mathbf{P}}\{\omega_1\} = \bar{p} = \Big(\frac{1}{S^d} - \frac{1}{(1+r)S_0}\Big)\,\frac{S^u S^d}{S^u - S^d} \tag{17}$$

and

$$\bar{\mathbf{P}}\{\omega_2\} = \bar{q} = \Big(\frac{1}{S^u} - \frac{1}{(1+r)S_0}\Big)\,\frac{S^u S^d}{S^d - S^u}\,. \tag{18}$$

It is easy to show that the properly modified version of the risk-neutral valuation formula has the following form

$$\pi_0(X) = S_0\, \mathbf{E}_{\bar{\mathbf{P}}}\big(S_T^{-1} X\big), \tag{19}$$

where X is a contingent claim which settles at time T. It appears that in some circumstances the choice of the stock price as a numeraire is more convenient than that of the savings account.

Let us apply this approach to the call option of Example 1.1. One finds easily that $\bar{p} = 0.62$, and thus formula (19) gives

$$\hat{C}_0 = S_0\, \mathbf{E}_{\bar{\mathbf{P}}}\big(S_T^{-1}\,(S_T - K)^+\big) = 21.59 = C_0,$$

as expected.

1.6 Put Option

We refer once again to Example 1.1. However, we shall now focus on a European put option instead of a call option. Since the buyer of a put option has the right to sell a stock at a given date T, the terminal payoff from the option is now $P_T = (K - S_T)^+$, i.e.,

$$P_T(\omega) = \begin{cases} P^u = 0, & \text{if } \omega = \omega_1, \\ P^d = 20, & \text{if } \omega = \omega_2, \end{cases}$$

where we have taken, as before, $K=\$280$. The portfolio $\phi = (\alpha_0, \beta_0)$ which replicates the European put option is thus determined by the following system of linear equations

$$\begin{cases} 320\,\alpha_0 + 1.05\,\beta_0 = 0, \\ 260\,\alpha_0 + 1.05\,\beta_0 = 20, \end{cases}$$

so that $\alpha_0 = -1/3$ and $\beta_0 = 101.59$. Consequently, the arbitrage price P_0 of the European put option equals

$$P_0 = -(1/3) \times 280 + 101.59 = 8.25.$$

Notice that the number of shares in a replicating portfolio is negative. This means that an option writer who wishes to hedge risk exposure should sell short at time 0 the number $-\alpha_0 = 1/3$ shares of stock for each sold put option. The proceeds from the short-selling of shares, as well as the option's premium, are invested in an interest-earning account. To find the arbitrage price of the put option we may alternatively apply Proposition 1.2. By virtue of (14), with $X = P_T$, we get

$$P_0 = \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}P_T\big) = 8.25.$$

Finally, the put option value can also be found by applying the following relationship between the prices of call and put options.

Corollary 1.1 *The following put-call parity relationship is valid*

$$C_0 - P_0 = S_0 - (1+r)^{-1}K. \tag{20}$$

Proof. The formula is an immediate consequence of equality (3) and the pricing formula (14) applied to the claim $S_T - K$. □

It is worthwhile to mention that relationship (20) is universal – that is, it does not depend on the choice of the model (the only assumption we need to make is the additivity of the price). Using the put-call parity, we can calculate once again the arbitrage price of the put option. Formula (20) yields immediately

$$P_0 = C_0 - S_0 + (1+r)^{-1}K = 8.25.$$

For ease of further reference, we shall write down explicit formulae for the call and put price in the one-period, two-state model. We assume, as usual, that $S^u > K > S^d$. Then

$$C_0 = \frac{S_0(1+r) - S^d}{S^u - S^d}\,\frac{S^u - K}{1+r}, \tag{21}$$

and

$$P_0 = \frac{S^u - S_0(1+r)}{S^u - S^d}\,\frac{K - S^d}{1+r}. \tag{22}$$

2 Futures Call and Put Options

We will first describe very succinctly the main features of futures contracts, which are reflected in stochastic models of futures markets to be developed later. As in the previous section, we will focus mainly on the arbitrage pricing of European call and put options; clearly, instead of the spot price of the underlying asset, we will now consider its futures price. The model of futures prices we adopt here is quite similar to the one used to describe spot prices. Still, due to the specific features of futures contracts used to set up a replicating strategy, one has to modify significantly the way in which the payoff from a portfolio is defined.

2.1 Futures Contracts and Futures Prices

A *futures contract* is an agreement to buy or sell an asset at a certain date in the future for a certain price. The important feature of these contracts is that they are traded on exchanges. Consequently, the authorities need to define precisely all the characteristics of each futures contract in order to make trading possible. More importantly, the *futures price* – the price at which a given futures contract is entered into – is determined on a given futures exchange by the usual law of demand and supply (in a similar way as for spot prices of listed stocks). Futures prices are therefore settled daily and the quotations are reported in the financial press. A futures contract is referred to by its delivery month, however an exchange specifies the period within that month when delivery must be made. The exchange specifies the amount of the asset to be delivered for one contract, as well as some additional details when necessary (e.g., the quality of a given commodity or the maturity of a bond). From our perspective, the most fundamental feature of a futures contract is the way the contract is settled. The procedure of daily settlement of futures contracts is called *marking to market.* A futures contract is worth zero when it is entered into; however, each investor is required to deposit funds into a *margin account.* The amount that should be deposited when the contract is entered into is known as the *initial margin.* At the end of each trading day, the balance of the investor's margin account is adjusted in a way that reflects daily movements of futures prices. To be more specific, if an investor assumes a long position, and on a given day the futures price rises, the balance of the margin account will also increase. Conversely, the balance of the margin account of any party with a short position in this futures contract will be properly reduced. Intuitively, it is thus possible to argue that futures contracts are actually closed out after each trading day, and then start afresh the next trading day. Obviously, to offset a position in a futures contract, an investor enters into the opposite trade to the original one. Finally, if the delivery period is reached, the delivery is made by the party with a short position.

2.2 One-period Futures Market

It will be convenient to start this section with a simple example which, in fact, is a straightforward modification of Example 1.1 to a futures market.

Example 2.1 Let $f_t = f_S(t, T^*)$ be a one-period process which models the futures price of a certain asset S, for the settlement date $T^* \geq T$. We assume that $f_0 = 280$, and

$$f_T(\omega) = \begin{cases} f^u = 320, & \text{if } \omega = \omega_1, \\ f^d = 260, & \text{if } \omega = \omega_2, \end{cases}$$

where $T = 3$ months.[3] We consider a 3-month European futures call option with strike price $K = \$280$. As before, we assume that the simple risk-free interest rate for 3-month deposits and loans is $r = 5\%$.

The payoff from the futures call option $C_T^f = (f_T - K)^+$ equals

$$C_T^f(\omega) = \begin{cases} C^{fu} = 40, & \text{if } \omega = \omega_1, \\ C^{fd} = 0, & \text{if } \omega = \omega_2. \end{cases}$$

A portfolio ϕ which replicates the option is composed of α_0 futures contracts and β_0 units of cash invested in riskless bonds (or borrowed). The wealth process $V_t^f(\phi)$, $t \in \{0, T\}$, of this portfolio equals $V_0^f(\phi) = \beta_0$, since futures contracts are worthless when they are first entered into. Furthermore, the terminal wealth of ϕ is

$$V_T^f(\phi) = \alpha_0 \,(f_T - f_0) + (1 + r)\beta_0, \tag{23}$$

where the first term on the right-hand side represents gains (or losses) from the futures contract, and the second corresponds to a savings account (or loan). Note that (23) reflects the fact that futures contracts are marked to market daily (that is, after each period in our model). A portfolio $\phi = (\alpha_0, \beta_0)$ is said to replicate the option when $V_T^f = C_T^f$, or more explicitly, if the equalities

$$V_T^f(\omega) = \begin{cases} \alpha_0(f^u - f_0) + (1 + r)\beta_0 = C^{fu}, & \text{if } \omega = \omega_1, \\ \alpha_0(f^d - f_0) + (1 + r)\beta_0 = C^{fd}, & \text{if } \omega = \omega_2 \end{cases}$$

[3] Notice that in the present context, the knowledge of the settlement date T^* of a futures contract is not essential. It is sufficient to assume that $T^* \geq T$.

are satisfied. For Example 2.1, this gives the following system of linear equations

$$\begin{cases} 40\,\alpha_0 + 1.05\,\beta_0 = 40, \\ -20\,\alpha_0 + 1.05\,\beta_0 = 0, \end{cases}$$

yielding $\alpha_0 = 2/3$ and $\beta_0 = 12.70$. The manufacturing cost of a futures call option is thus $C_0^f = V_0^f(\phi) = \beta_0 = 12.70$. Similarly, the unique portfolio replicating a sold put option is determined by the following conditions

$$\begin{cases} 40\,\alpha_0 + 1.05\,\beta_0 = 0, \\ -20\,\alpha_0 + 1.05\,\beta_0 = 20, \end{cases}$$

so that $\alpha_0 = -1/3$ and $\beta_0 = 12.70$ in this case. Consequently, the manufacturing costs of put and call futures options are equal in our example. As we shall see soon, this is not a pure coincidence; in fact, by virtue of formula (29) below, the prices of call and put futures options are equal when the option's strike price coincides with the initial futures price of the underlying asset. The above considerations may be summarized by means of the following exhibits (note that β_0 is a positive number)

$$\text{at time } t=0 \quad \begin{cases} \text{one sold futures option} & C_0^f, \\ \text{futures contracts} & 0, \\ \text{cash deposited in a bank} & -\beta_0 = -C_0^f, \end{cases}$$

and

$$\text{at time } t=T \quad \begin{cases} \text{option's payoff} & -C_T^f, \\ \text{profits/losses from futures} & \alpha_0\,(f_T - f_0), \\ \text{cash withdrawal} & \hat{r}\beta_0, \end{cases}$$

where, as before, $\hat{r} = 1 + r$.

2.3 Martingale Measure for a Futures Market

We are looking now for a probability measure $\tilde{\mathbf{P}}$ which makes the futures price process (with no discounting) follow a $\tilde{\mathbf{P}}$-martingale. A probability $\tilde{\mathbf{P}}$, if it exists, is thus determined by the equality

$$f_0 = \mathbf{E}_{\tilde{\mathbf{P}}}\big(f_T\big) = \tilde{p}\, f^u + (1-\tilde{p})\, f^d. \tag{24}$$

It is easily seen that

$$\tilde{\mathbf{P}}\{\omega_1\} = \tilde{p} = \frac{f_0 - f^d}{f^u - f^d}, \quad \tilde{\mathbf{P}}\{\omega_2\} = 1 - \tilde{p} = \frac{f_u - f_0}{f^u - f^d}\,. \tag{25}$$

Using the data of Example 2.1, one finds easily that $\tilde{p} = 1/3$. Consequently, the expected value under the probability $\tilde{\mathbf{P}}$ of the discounted payoff from the futures call option equals

$$\tilde{C}_0^f = \mathbf{E}_{\tilde{\mathbf{P}}}\big((1+r)^{-1}(f_T - K)^+\big) = 12.70 = C_0^f.$$

This illustrates the fact that the martingale approach may be used also in the case of futures markets, with a suitable modification of the notion of a martingale measure.

Remarks. Using the traditional terminology of mathematical finance, we may conclude that the risk-neutral futures economy is characterized by the fair-game property of the process of a futures price. Remember that the risk-neutral spot economy is the one in which the discounted stock price (as opposed to the stock price itself) models a fair game.

2.4 Absence of Arbitrage

In this subsection, we shall study a general two-state, one-period model of a futures price. We consider the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\{0,T\}}, \mathbf{P})$ introduced in Sect. 1.4. The first process, which intends to model the dynamics of the futures price of a certain asset for the fixed settlement date $T^* \geq T$, is an adapted and strictly positive process $f_t = f_S(t, T^*)$, $t = 0, T$. More specifically, f_0 is assumed to be a real number, and f_T is the following random variable

$$f_T(\omega) = \begin{cases} f^u, & \text{if } \omega = \omega_1, \\ f^d, & \text{if } \omega = \omega_2, \end{cases}$$

where, by convention, $f^u > f^d$. The second security is, as in the case of a spot market, a riskless bond whose price process is $B_0 = 1$, $B_T = 1 + r$ for some real $r \geq 0$. Let Φ^f stand for the linear space of all futures contracts-bonds portfolios $\phi = \phi_0 = (\alpha_0, \beta_0)$; it may be, of course, identified with the linear space $\mathbf{R}^2$. The wealth process $V^f(\phi)$ of any portfolio equals

$$V_0(\phi) = \beta_0, \quad \text{and} \quad V_T^f(\phi) = \alpha_0(f_T - f_0) + (1+r)\beta_0 \tag{26}$$

(it is useful to compare these formulae with (4)). We shall study the valuation of derivatives in the futures market model $\mathcal{M}^f = (f, B, \Phi^f)$. It is easily seen that an arbitrary contingent claim X which settles at time T admits a unique replicating portfolio $\phi \in \Phi^f$. Put another way, all contingent claims which settle at time T are *attainable* in the market model $\mathcal{M}^f$. In fact, if X is given by the formula

$$X(\omega) = \begin{cases} X^u & \text{if } \omega = \omega_1, \\ X^d & \text{if } \omega = \omega_2, \end{cases}$$

then its replicating portfolio $\phi \in \Phi^f$ may be found by solving the following system of linear equations

$$\begin{cases} \alpha_0(f^u - f_0) + (1+r)\beta_0 = X^u, \\ \alpha_0(f^d - f_0) + (1+r)\beta_0 = X^d. \end{cases} \tag{27}$$

The unique solution of (27) is

$$\alpha_0 = \frac{X^u - X^d}{f^u - f^d}, \quad \beta_0 = \frac{X^u(f_0 - f^d) + X^d(f^u - f_0)}{(1+r)(f^u - f^d)}. \tag{28}$$

Consequently, the *manufacturing cost* $\pi_0^f(X)$ in $\mathcal{M}^f$ equals

$$\pi_0^f(X) \stackrel{\text{def}}{=} V_0^f(\phi) = \beta_0 = \frac{X^u(f_0 - f^d) + X^d(f^u - f_0)}{(1+r)(f^u - f^d)}. \tag{29}$$

We say that a model $\mathcal{M}^f$ of the futures market is *arbitrage-free* if there are no arbitrage opportunities in the class Φ^f of trading strategies. The following simple result provides necessary and sufficient conditions for the arbitrage-free property of $\mathcal{M}^f$.

Proposition 2.1 *The futures market $\mathcal{M}^f = (f, B, \Phi^f)$ is arbitrage-free if and only if the process f that models the futures price admits a (unique) martingale measure $\tilde{\mathbf{P}}$ equivalent to $\mathbf{P}$. In this case, the arbitrage price at time 0 of any contingent claim X which settles at time T equals*

$$\pi_0^f(X) = \mathbf{E}_{\tilde{\mathbf{P}}}\big((1+r)^{-1}X\big), \tag{30}$$

or explicitly

$$\pi_0^f(X) = \frac{f_0 - f^d}{f^u - f^d}\,\frac{X^u}{1+r} + \frac{f^u - f_0}{f^u - f^d}\,\frac{X^d}{1+r}. \tag{31}$$

Proof. If there is no martingale measure for f which is equivalent to $\mathbf{P}$, we have either $\tilde{p} \geq 1$ or $\tilde{p} \leq 0$. In the first case, we have $f_0 - f^d \geq f^u - f^d$ and thus $f_0 \geq f^u > f^d$. Consequently, a portfolio $\phi = (-1, 0)$ is an arbitrage opportunity. Similarly, when $\tilde{p} \leq 0$ the inequalities $f_0 \leq f^d < f^u$ are valid. Therefore the portfolio $\phi = (1, 0)$ is an arbitrage opportunity. Finally, if $0 < \tilde{p} < 1$ and for some $\phi \in \Phi^f$ we have $V_0^f(\phi) = 0$, then it follows from (29) that

$$\frac{f_0 - f^d}{f^u - f^d}\, V^{fu} + \frac{f^u - f_0}{f^u - f^d}\, V^{fd} = 0$$

so that $V^{fd} < 0$ if $V^{fu} > 0$, and $V^{fu} < 0$ when $V^{fd} > 0$. This shows that the market model $\mathcal{M}^f$ is arbitrage-free if and only if the process f admits a martingale measure equivalent to $\mathbf{P}$. The valuation formula (30) now follows by (25)–(29). □

When the price of the futures call option is already known, in order to find the price of the corresponding put option one may use the following relation, which is an immediate consequence of equality (3) and the pricing formula (30)

$$C_0^f - P_0^f = (1 + r)^{-1}(f_0 - K). \tag{32}$$

It is now obvious that the equality $C_0^f = P_0^f$ is valid if and only if $f_0 = K$; that is, when the current futures price and the strike price of the option are equal. Equality (32) is referred to as the *put-call parity relationship* for futures options.

2.5 One-period Spot/Futures Market

Consider an arbitrage-free, one-period spot market (S, B, Φ) described in Sect. 1. Moreover, let $f_t = f_S(t, T), t \in \{0, T\}$ be the process of futures prices with the underlying asset S and for the maturity date T. In order to preserve consistency with the financial interpretation of the futures price, we have to assume that $f_T = S_T$. Our aim is to find the right value f_0 of the futures price at time 0; that is, that level of the price f_0 which excludes arbitrage opportunities in the combined spot/futures market. In such a market, trading in stocks, bonds, as well as entering into futures contracts is allowed.

Corollary 2.1 *The futures price at time 0 for the delivery date T of the underlying asset S which makes the spot/futures market arbitrage-free equals $f_0 = (1 + r)S_0$.*

Proof. Suppose an investor enters at time 0 into one futures contract. The payoff of his position at time T corresponds to a time T contingent claim $X = f_T - f_0 = S_T - f_0$. Since it costs nothing to enter a futures contract we should have

$$\pi_0(X) = \pi_0(S_T - f_0) = 0,$$

or equivalently

$$\pi_0(X) = S_t - (1 + r)^{-1} f_0 = 0.$$

This proves the asserted formula. Alternatively, one can check that if the futures price f_0 were different from $(1 + r)S_0$, this would lead to arbitrage opportunities in the spot/futures market. □

3 Forward Contracts

A *forward contract* is an agreement, signed at the initial date 0, to buy or sell an asset at a certain future time T (called *delivery date* or *maturity* in what follows) for a prespecified price K, referred to as the *delivery price*. In contrast to stock options and futures contracts, forward contracts are not traded on exchanges. By convention, the party who agrees to buy the underlying asset at time T for the delivery price K is said to assume a *long position* in a given contract. Consequently, the other party, who is obliged to sell the asset at the same date for the price K, is said to assume a *short position.* Since a forward contract is settled at maturity and a party in a long position is obliged to buy an asset worth S_T at maturity for K, it is clear that the payoff from the long position (from the short position, respectively) in a given forward contract with a stock S being the underlying asset corresponds to the time T contingent claim X ($-X$, respectively), where

$$X = S_T - K. \tag{33}$$

It should be emphasized that there is no cash flow at the time the forward contract is entered into. In other words, the price (or value) of a forward contract at its initiation is zero. Notice, however, that for $t > 0$, the value of a forward contract may be negative or positive. As we shall now see, a forward contract is worthless at time 0 provided that a judicious choice of the delivery price K is made.

Before we end this section, we shall find the rational delivery price for a forward contract. To this end, let us introduce first the following definition which is, of course, consistent with typical features of a forward contract. Recall that, typically, there is no cash flow at the initiation of a forward contract.

Definition 3.1 The delivery price K that makes a forward contract worthless at initiation is called the *forward price* of an underlying financial asset S for the settlement date T.

Note that we use here the adjective *financial* in order to emphasize that the storage costs, which have to be taken into account when studying forward contracts on commodities, are neglected. In the case of a dividend-paying stock, in the calculation of the forward price, it is enough to substitute S_0 with $S_0 - \hat{I}_0$, where $\hat{I}_0$ is the present value of all future dividend payments during the contract's lifetime.

Proposition 3.1 *Assume that the one-period, two-state security market model (S, B, Φ) is arbitrage-free. Then the forward price at time 0 for the settlement date T of one share of stock S equals $F_S(0,T) = (1+r)S_0$.*

Proof. We shall apply the martingale method of Proposition 1.2. By applying formulae (14) and (33), we get

$$\pi_0(X) = \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-1}X\big) = \mathbf{E}_{\mathbf{P}^*}(S_T^*) - \hat{r}^{-1}K = S_0 - \hat{r}^{-1}K = 0, \tag{34}$$

where $\hat{r} = 1 + r$. It is now apparent that $F_S(0,T) = (1+r)S_0$. □

By combining Corollary 2.1 with the above proposition, we conclude that in a one-period model of a spot market, the futures and forward prices of financial assets for the same settlement date are equal.

4 Options of American Style

An option of *American style* (or briefly, an *American option*) is an option contract in which not only the decision whether to exercise the option or not, but also the choice of the exercise time, is at the discretion of the option's holder. The exercise time cannot be chosen after the option's expiry date T. Hence, in our simple one-period model, the strike price can either coincide with the initial date 0, or with the terminal date T. Notice that the value (or the price) at the terminal date of the American call or put option written on any asset equals the value of the corresponding European option with the same strike price K. Therefore, the only unknown quantity is the price of the American option at time 0. In view of the early exercise feature of the American option, the concept of perfect replication of the terminal option's payoff is not adequate for valuation purposes. To determine this value, we shall make use of the general rule of absence of arbitrage in the market model. By definition, the arbitrage price at time 0 of the American option should be set in such a way that trading in American options would not destroy the arbitrage-free feature the market.

We will first show that the American call written on a stock that pays no dividends during the option's lifetime is always equivalent to the European call; that is, that both options necessarily have identical prices at time 0. As we shall see in what follows, such a property is not always true in the case of American put options; that is, American and European puts are not equivalent, in general.

We place ourselves once again within the framework of a one-period spot market $\mathcal{M} = (S, B, \Phi)$, as specified in Sect. 1.1. It will be convenient to assume that European options are traded securities in our market. For $t = 0, T$, let us denote by C_t^a and P_t^a the arbitrage price at time t of the American call and put, respectively. It is obvious that $C_T^a = C_T$ and $P_T^a = P_T$. As mentioned earlier, both arbitrage prices C_0^a and P_0^a will be determined using the following property: if the market $\mathcal{M} = (S, B, \Phi)$ is arbitrage-free, then the market with trading in stocks, bonds and American options should remain arbitrage-free. It should be noted that it is not evident a priori that the last property determines in a unique way the values of C_0^a and P_0^a. We assume throughout that the inequalities $S^d < S_0(1+r) < S^u$ hold and the strike price satisfies $S^d < K < S^u$. Otherwise, either the market model would not be arbitrage-free, or valuation of the option would be a trivial matter.

Proposition 4.1 *Assume that the risk-free interest rate r is a non-negative real number. Then the arbitrage price C_0^a of an American call option in the arbitrage-free market model $\mathcal{M} = (S, B, \Phi)$ coincides with the price C_0 of the European call option with the same strike price K.*

Proof. Assume, on the contrary, that $C_0^a \neq C_0$. Suppose first that $C_0^a > C_0$. Notice that the arbitrage price C_0 satisfies

$$C_0 = p_* \frac{S^u - K}{1+r} = \frac{(1+r)S_0 - S^d}{S^u - S^d} \frac{S^u - K}{1+r} > S_0 - K, \tag{35}$$

if $r \geq 0$. It is now straightforward to check that there exists an arbitrage opportunity in the market. In fact, to create a riskless profit, it is sufficient to sell the American call option at C_0^a, and simultaneously buy the European call option at C_0. If European options are not traded, one may, of course, create a replicating portfolio for the European call at initial investment C_0. The above portfolio is easily seen to lead to a riskless profit, independently from the decision regarding the exercise time made by the holder of the American call. If, on the contrary, the price C_0^a were strictly smaller than C_0, then by selling European calls and buying American calls, one would be able to create a profitable riskless portfolio. □

It is worthwhile to observe that inequality (35) is valid in a more general setup. Indeed, if $r \geq 0$, $S_0 > K$, and S_T is a $\mathbf{P}^*$-integrable random variable, then we have always

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}(S_T - K)^+) &\geq \Big(\mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}S_T) - (1+r)^{-1}K\Big)^+ \\ &= (S_0 - (1+r)^{-1}K)^+ \geq S_0 - K, \end{aligned}$$

where the first inequality follows by Jensen's inequality. Notice that in the case of the put option we get merely

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}(K - S_T)^+) &\geq \Big(\mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}K - (1+r)^{-1}S_T)\Big)^+ \\ &= ((1+r)^{-1}K - S_0)^+ > K - S_0, \end{aligned}$$

where the last inequality holds provided that $-1 < r < 0$. If $r = 0$, we obtain

$$\mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}(K - S_T)^+) = K - S_0.$$

Finally, if $r > 0$, no obvious relationship between P_0 and $S_0 - K$ is available. This feature suggests that the counterpart of Proposition 4.1 – the case of American put – should be more interesting.

Proposition 4.2 *Assume that $r > 0$. Then $P_0^a = P_0$ if and only if the inequality*

$$K - S_0 \leq \frac{S^u - (1+r)S_0}{S^u - S^d} \frac{K - S^d}{1+r} \tag{36}$$

is valid. Otherwise, $P_0^a = K - S_0 > P_0$. If $r = 0$, then invariably $P_0^a = P_0$.

Proof. In view of (22), it is clear that inequality (36) is equivalent to $P_0 \geq K - S_0$. Suppose first that the last inequality holds. If, in addition, $P_0^a > P_0$ ($P_0^a < P_0$, respectively), by selling the American put and buying the European put (by buying the American put and selling the European put, respectively) one creates a profitable riskless strategy. Hence, $P_0^a = P_0$ in this case.[4] Suppose now that (36) fails to hold – that is, $P_0 < K - S_0$, and assume that $P_0^a \neq K - S_0$. We wish to show that P_0^a should be set to be $K - S_0$, otherwise arbitrage opportunities arise. Actually, if P_0^a were strictly greater that $K - S_0$, the seller of an American put would be able to lock in a profit by perfectly hedging exposure using the European put acquired at a strictly lower cost P_0. If, on the contrary, inequality $P_0^a < K - S_0$ were true, it would be profitable to buy the American put and exercise it immediately. Summarizing, if (36) fails to hold, the arbitrage price of the American put is strictly greater than the price of the European put. Finally, one verifies easily that if the holder of the American put fails to exercise it at time 0, the option's writer is still able to lock in a profit. Hence, if (36) fails to hold, the American put should be exercised immediately, otherwise arbitrage opportunities would arise in the market. For the last statement, observe that if $r = 0$, then inequality (36), which now reads

$$K - S_0 \leq \frac{S^u - S_0}{S^u - S^d} (K - S^d),$$

is easily seen to be valid (it is enough to take $K = S^d$ and $K = S^u$). □

The above results suggest the following general "rational" exercise rule in a discrete-time framework: at any time t before the option's expiry, find the maximal expected payoff over all admissible exercise rules and compare the outcome with the payoff obtained by exercising the option immediately. If the latter value is greater, exercise the option immediately, otherwise go one step further. In fact, one checks easily that the price at time 0 of an American call or put option may be computed as the maximum expected value of the payoff over all exercises, provided that the expectation in question is taken under the martingale probability measure. The last feature distinguishes arbitrage pricing of American options from the typical optimal stopping problems, in which maximization of expected payoffs takes place under a subjective (or actual) probability measure rather than under an artificial martingale measure. We conclude that a simple argument that the rational option's holder will always try to maximize the expected payoff of the option at exercise is not sufficient to determine arbitrage prices of American claims. A more precise statement would read: the American put option should be exercised by its holder at the same date as it is exercised by a risk-neutral individual whose objective is to maximize the discounted expected payoff of the option; otherwise arbitrage opportunities would arise in the market. It will be useful to formalize the concept of an *American contingent claim.*

Definition 4.1 A contingent claim of American style (or shortly, *American claim*) is a pair $X^a = (X_0, X_T)$, where X_0 is a real number and X_T is a random variable. We interpret X_0 and X_T as the payoffs received by the holder of the American claim X^a if he chooses to exercise it at time 0 and at time T, respectively.

Notice that in our present setup, the only admissible *exercise times* are the initial date and the expiry date, say $\tau_0 = 0$ and $\tau_1 = T$. By convention, we say that an option is exercised at expiry date T if it is not exercised prior to that date, even when its terminal payoff equals zero (so that in fact the option is abandoned). We assume also, for simplicity, that $T = 1$. Then we may formulate the following corollary to Propositions 4.1–4.2, whose proof is left as exercise.

[4] To be formal, we need to check that no arbitrage opportunities are present if $P_0^a = P_0$ and (36) holds.

Corollary 4.1 *The arbitrage prices of an American call and an American put option in the arbitrage-free market model $\mathcal{M} = (S, B, \Phi)$ are given by*

$$C_0^a = \max_{\tau \in \mathcal{T}} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-\tau}(S_\tau - K)^+\big)$$

and

$$P_0^a = \max_{\tau \in \mathcal{T}} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-\tau}(K - S_\tau)^+\big)$$

respectively, where $\mathcal{T}$ denotes the class of all exercise times. More generally, if $X^a = (X_0, X_T)$ is an arbitrary contingent claim of American style, then its arbitrage price $\pi(X^a)$ in $\mathcal{M} = (S, B, \Phi)$ equals

$$\pi_0(X^a) = \max_{\tau \in \mathcal{T}} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-\tau} X_\tau\big), \quad \pi_T(X^a) = X_T.$$

4.1 Universal No-arbitrage Inequalities

We shall now derive universal inequalities that are necessary for absence of arbitrage in the market. It is clear that the following property is valid in any discrete- or continuous-time, arbitrage-free market.

Price monotonicity rule. In any model of an arbitrage-free market, if X_T and Y_T are two European contingent claims, where $X_T \geq Y_T$, then $\pi_t(X_T) \geq \pi_t(Y_T)$ for every $t \in [0, T]$, where $\pi_t(X_T)$ and $\pi_t(Y_T)$ denote the arbitrage prices at time t of X_T and Y_T, respectively. Moreover, if $X_T > Y_T$, then $\pi_t(X_T) > \pi_t(Y_T)$ for every $t \in [0, T]$.

For the sake of notational convenience, a constant (non-negative) rate r will now be interpreted as a continuously compounded rate of interest. Hence, the price at time t of one dollar to be received at time $T \geq t$ equals $e^{-r(T-t)}$; in other words, the savings account process equals $B_t = e^{rt}$ for every $t \in [0, T]$.

Proposition 4.3 *Let C_t and P_t (C_t^a and P_t^a, respectively) stand for the arbitrage prices at time t of European (American, respectively) call and put options, with strike price K and expiry date T. Then the following inequalities are valid for every $t \in [0, T]$*

$$(S_t - Ke^{-r(T-t)})^+ \leq C_t = C_t^a \leq S_t, \tag{37}$$

$$(Ke^{-r(T-t)} - S_t)^+ \leq P_t \leq K, \tag{38}$$

and

$$(K - S_t)^+ \leq P_t^a \leq K. \tag{39}$$

The put-call parity relationship, which in the case of European options reads

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \tag{40}$$

takes, in the case of American options, the form of the following inequalities

$$S_t - K \leq C_t^a - P_t^a \leq S_t - Ke^{-r(T-t)}. \tag{41}$$

Proof. All inequalities may be derived by constructing appropriate portfolios at time t and holding them to the terminal date. Let us consider, for instance, the first one. Consider the following portfolios, A and B. Portfolio A consists of one European call and $Ke^{-r(T-t)}$ of cash; portfolio B contains only one share of stock. The value of the first portfolio at time T equals

$$C_T + K = (S_T - K)^+ + K = \max\{S_T, K\} \geq S_T,$$

while the value of portfolio B is exactly S_T. Hence, the arbitrage price of portfolio A at time t dominates the price of portfolio B – that is,

$$C_t + Ke^{-r(T-t)} \geq S_t, \quad \forall\, t \in [0, T].$$

Since the price of the option is non-negative, this proves the first inequality in (37). All remaining inequalities in (37)–(39) may be verified by means of similar arguments. To check that $C_t^a = C_t$, we consider the following portfolios: portfolio A – one American call option and $Ke^{-r(T-t)}$ of cash; and portfolio B – one share of stock. If the call option is exercised at some date $t^* \in [t, T]$, then the value of portfolio A at time t^* equals

$$S_{t^*} - K + Ke^{-r(T-t^*)} < S_{t^*},$$

while the value of B is S_{t^*}. On the other hand, the value of portfolio A at the terminal date T is $\max\{S_T, K\}$, hence it dominates the value of portfolio B, which is S_T. This means that early exercise of the call option would contradict our general price monotonicity rule. A justification of relationship (40) is straightforward, as $C_T - P_T = S_T - K$. To justify the second inequality in (41), notice that in view of (40) and the obvious inequality $P_t^a \geq P_t$, we get

$$P_t^a \geq P_t = C_t^a + Ke^{-r(T-t)} - S_t, \quad \forall\, t \in [0, T].$$

The proof of the first inequality in (41) goes as follows. Take the two following portfolios: portfolio A – one American call and K units of cash; and portfolio B – one American put and one share of stock. If the put option is exercised at time $t^* \in [t, T]$, then the value of portfolio B at time t^* is K. On the other hand, the value of portfolio A at this date equals

$$C_t + Ke^{r(t^*-t)} \geq K.$$

Therefore, portfolio A is more valuable at time t than portfolio B; that is

$$C_t^a + K \geq P_t^a + S_t$$

for every $t \in [0, T]$. □

Let us denote by $C(S_0, T, K)$ ($C^a(S_0, T, K)$, respectively) the price of the European (American, respectively) call option with expiration date T and exercise price K. The following relationships are easy to derive

$$C^a(S_0, T_1, K) \leq C^a(S_0, T_2, K)$$

where $T_1 \leq T_2$, and

$$C(S_0, T, K_2) \leq C(S_0, T, K_1), \quad C^a(S_0, T, K_2) \leq C^a(S_0, T, K_1)$$

provided that $K_1 \leq K_2$.

Proposition 4.4 *Assume that $K_1 < K_2$. The following inequalities are valid*

$$e^{-rT}(K_1 - K_2) \leq C(S_0, T, K_2) - C(S_0, T, K_1) \leq 0,$$

and

$$K_1 - K_2 \leq C^a(S_0, T, K_2) - C^a(S_0, T, K_1) \leq 0.$$

Proof. Let us consider, for instance, the case of European options. Take the two following portfolios at time 0: portfolio A – one European call with exercise price K_2 and $e^{-rT}(K_2 - K_1)$ units of cash; and portfolio B – one European call with exercise price K_1. The value of portfolio A at time T is

$$(S_T - K_2)^+ + (K_2 - K_1) \geq (S_T - K_1)^+,$$

and the value of portfolio B at time T equals $(S_0 - K_1)^+$. Consequently

$$C(S_0, T, K_2) + e^{-rT}(K_2 - K_1) \geq C(S_0, T, K_1),$$

as expected. □

Proposition 4.5 *The price of a European (or American) call (or put) option is a convex function of the exercise price K.*

Proof. Let us consider the case of a European put option. We write $P(S_0, T, K)$ to denote its price at time 0. Assume that $K_1 < K_2$ and put $K_3 = \gamma K_1 + (1-\gamma)K_2$, where $\gamma \in (0,1)$ is a constant. We consider the following portfolios: portfolio A – γ European put options with exercise price K_1 and $1-\gamma$ European put options with exercise price K_2; and portfolio B – one European put option with exercise price K_3. At maturity T we have

$$\gamma(K_1 - S_T)^+ + (1-\gamma)(K_2 - S_T)^+ \geq \Big(\big(\gamma K_1 + (1-\gamma)K_2\big) - S_T\Big)^+$$

since the payoff function $h(x) = (K-x)^+$ is convex. $\square$

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